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Mathematical simulation of wave fields in media with arbitrary curvilinear boundaries

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Abstract

The method of calculation of wave fields in media with curvilinear interfaces is proposed. After integral transformations, the solution in the explicit form is obtained. For solving the problem, we introduce the notion of the generalized thickness of a layer. This value becomes real in the case of a plane boundary and coincides with an "ordinary" thickness of a layer. If the boundary is curvilinear, the generalized thickness of a layer becomes a complex value. The use of the generalized thickness of a layer brings about the fact that conditions on infinity (Sommerfeld's conditions) are automatically fulfilled. The conditions of appearance of loops on the travel-time curves for different boundaries have been elucidated. Some examples of calculations are given.

Keywords: Mathematical simulation; Wave fields; Curvilinear boundaries; Multidimensional media

1. Introduction

In problems of simulation of wave fields, geophysical fields in particular, of primary importance is the choice of tests allowing the control of accuracy of algorithms used. Currently, there are not many exact solutions for multidimensional media [1]. In rare cases, in which the exact problem solution is known, it has, as a rule, a complicated shape, consisting of series whose each term of which is represented as integral. This paper proposes an analytical method of calculation of wave fields in media with arbitrary curvilinear boundaries. The problem in question has also an independent meaning, for example, in the oil

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seismic prospecting. In this case, the analytical form of the method proposed is characterized of a relative simplicity. The solution to the given problem is built by means of introduction of a complex generalized thickness of a layer. This makes the presence of diffraction components of the wave field physically visible. However, when introducing complex values, the Sommerfeld conditions are automatically taken into account.

2. Statement of the problem

In the Cartesian coordinate system, the problem is formulated in the following way: define the coordinates of the displacement vector for the anisotropic non-elastic medium, satisfying the system of equations:

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2} + F_x f(t), \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho \frac{\partial^2 u_y}{\partial t^2} + F_y f(t), \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2} + F_y f(t),
\end{aligned} \tag{1}$$

with the initial conditions (for $t = 0$)

$$u_x = \frac{\partial u_x}{\partial t} = u_y = \frac{\partial u_y}{\partial t} = u_z = \frac{\partial u_z}{\partial t} = 0 \tag{2}$$

and the boundary conditions (for $z = 0$)

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0. \tag{3}$$

Components of the stress tensor are assumed to be connected with those of the deformation tensor by the relations [2]

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix}.$$

The deformation tensor components ε are conventionally defined by the displacement vector components [2]. The calculation of waves in anisotropy non-elastic media is based on the Volterra relations with allowance for the elastic after-effects, i.e. elastic constants, c_{ij} are replaced for the integral operators C_{ij} [3].

$$C_{ij} x \equiv c_{ij} x(t) - c_{ij}^1 \int_{-\infty}^{t} h_{ij}(t - \tau) x(\tau) \mathrm{d}\tau.$$

Here c_{ij}^1, h_{ij} are parameters, defining the absorption level. These parameters are assumed to satisfy the conditions:

1. h_{ij} are positive continuous and monotonic by decreasing for $t \to \infty$.
2. $c_{ij}^1 \int_0^\infty g(t)\mathrm{d}t < c_{ij}, \quad c_{ij}^1 > 0$.
3. $h_{ij|_{t<0}} \equiv 0, \quad t \cdot h_{ij}(t) \underset{t\to\infty}{\to} 0$.

Elastic and non-elastic parameters are assumed to be arbitrary piecewise-continuous functions of variables x, y, z, F_x, F_y, F_z are the force vector components, which describe concentrated and distributed seismic sources of various types. Note, that the initial problem statement is of the integro-differential form.

The basic steps of the algorithm (to simplify the formulas) are illustrated on a model of the wave equation in the Cartesian coordinate system:

$$\begin{aligned} &\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial z^2} = \frac{1}{c^2(x,z)} \frac{\partial^2 g}{\partial t^2}, \\ &g|_{t=0} = \left.\frac{\partial g}{\partial t}\right|_{t=0} = 0, \quad \left.\frac{\partial g}{\partial z}\right|_{z=0} = f(t)\delta(x - x_0). \end{aligned} \tag{4}$$

Here $f(t)$ is the temporal impulse.

The depth down the interface is set in the form: $Z = z(x)$. In this case, a depth variation occurs with respect to the free surface. The conditions of discontinuity should now be fulfilled at the discontinuity velocity boundary, which divides the semi-space into two domains. Due to this fact, the conventional conjugation conditions are introduced at the discontinuity boundaries.

3. Algorithm of solution

The desired solution is sought for as a Fourier transform in the variables t and x. Let ω and k be temporal and spatial frequencies, $\widehat{g}$ the corresponding image of the desired transformation. In what follows, the corresponding symbols and indices will be omitted for simplicity.

For constructing a solution in the spectral domain of problem (4), let us first consider the case of a plane boundary. The solution in this case will look like [4].

$$\begin{aligned} &g(z,k,\omega)|_{z=0} = \frac{1}{\nu_1} \frac{1 + q\mathrm{e}^{-2\nu_1 h}}{1 - q\mathrm{e}^{-2\nu_1 h}} \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t, \\ &\nu_1 = \sqrt{k^2 - \frac{\omega^2}{c_1^2}}, \quad \nu_2 = \sqrt{k^2 - \frac{\omega^2}{c_2^2}}, \quad q = \frac{\nu_2 - \nu_1}{\nu_2 + \nu_1}. \end{aligned} \tag{5}$$

Now we introduce a functional and find its extreme values.

$$d = \sqrt{z(x)^2 + (x - x_0)^2}. \tag{6}$$

It is easy to show that in the case of a plane boundary, the extreme value (6) will be equal to the corresponding layer thickness. This principle underlies the ways of finding the distance for media with curvilinear boundaries.

To find a solution to (4) for the curvilinear boundary we introduce a complex distance from (6) in the following manner: the value of the layer thickness h in (5) is replaced for the stationary value d in (6), which is generally complex.

$$h \to d = d_1 + \mathrm{i}d_2.$$

Considering in (5) only single waves, we obtain—according to the similarity principle—the solution of the problem in question

$$g = \frac{1}{\nu_1}\left(1 + \sum_{k=1}^{N} q \mathrm{e}^{-2\nu_1 d_k}\right). \tag{7}$$

Here N is the number of stationary points (6).

By way of example, let us first consider the case of the parabolic boundary:

$$z(x) = a(x - r_1)^2 + h. \tag{8}$$

In this case, extreme (6) will be defined from the following equation:

$$z^3 + pz + q. \tag{9}$$

Here the following relations are introduced:

$$p = \frac{2ab+1}{2a^2}, \quad q = \frac{r_1 - x}{2a^2}, \quad a = \frac{h - h_1}{2r_1^2}, \quad b = h_1, \quad D = \left(\frac{p}{2}\right)^3 + \left(\frac{q}{2}\right)^2,$$
$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}} + r_1 = l + m + r_1 = A + r_1,$$
$$z = a\left(\sqrt[3]{-\frac{q}{2} - \sqrt{D}} + \sqrt[3]{-\frac{q}{2} + \sqrt{D}}\right)^2 + b.$$

If $h = 0$, then for (6) we obtain:

$$d(x) = \sqrt{a^2 A^4 + (A + r_1 - x)^2}.$$

Eq. (9) has three real roots if

$$D = \left(\frac{p}{2}\right)^3 + \left(\frac{q}{2}\right)^2 < 0. \tag{10}$$

Inequality (10) takes rather a simple form for the case $r = r_1$. The condition for the three roots will then have the form:

$$h > \frac{\sqrt{2}}{2} r_1. \tag{11}$$

From (11) it follows that with an increase of the parabola sagging there will be one real root at first. Then with an increase of h there will be three. It is obvious that such a character of changing the roots will be observed on the whole observation area. This physically means that with an increase of the parabola sagging, there arises one loop on the travel-time curve. This is confirmed by the calculation presented in Fig. 1 (see below).

Then the following boundary is considered:

$$z(x) = -\frac{h}{r_1^4}(x - r_1)^4 + h. \tag{12}$$

In this case the critical points (6) are defined from the following equation.

$$z^7 + \frac{h}{c} z^3 + \frac{1}{4c^2} z + \frac{r_1 - x}{4c^2} = 0. \tag{13}$$

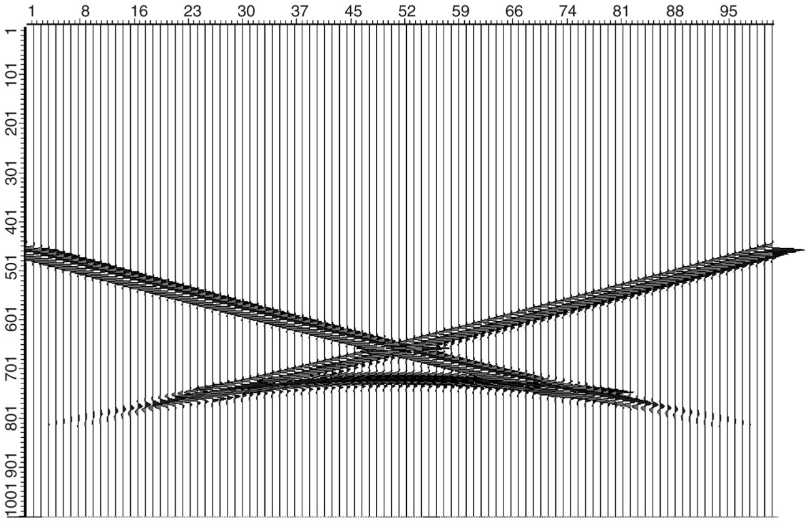

Fig. 1. Wave field for complex thickness of a layer.

Here the following notation is introduced: $c = -\frac{h}{r_1^4}$.

Eq. (13) has seven roots, among which one is always real. To simplify the analysis, we assume that $r = r_1$. In this case, Eq. (13) will take the following form:

$$z^6 + \frac{h}{c} z^2 + \frac{1}{4c^2} = 0. \tag{14}$$

Let $w = z^2$.

Then Eq. (14) will look like

$$w^3 + pw + q = 0. \tag{15}$$

Here we introduce the following notation:

$$p = -r_1^4, \quad q = \frac{r_1^8}{4h^2}.$$

Eq. (15) will have three real roots if its discriminate is lesser than zero. This brings about the following expression for the sagging (13).

$$h > \frac{\sqrt[4]{2}}{2} r_1. \tag{16}$$

Thus, when fulfilling condition (16), Eq. (15) will have three real roots. Now let us clear up when Eq. (15) has three positive roots. To this end, let us use the relation between the roots and coefficients (15)

$$\begin{aligned} &w_1 + w_2 + w_3 = 0, \\ &w_1 w_2 + w_1 w_3 + w_2 w_3 = p, \\ &w_1 w_2 w_3 = -q. \end{aligned} \tag{17}$$

Let us now make use of the Descartes theorem [5]:

The number of positive roots in the algebraic equation

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0, \quad a_0 \neq 0,$$

with allowance for their multiplicity is equal to the number of changes of signs in the system of coefficients

$$a_0, a_1, \ldots, a_n,$$

where the coefficients equal to zero are not taken into account.

Application of the above theorem yields the sequence

$$1, 0, -r_1^4, \frac{r_1^8}{4h^2}. \tag{18}$$

Sequence (18) has two changes of sign. Thus, Eq. (15) has two positive roots. Let us find out when they are multiple.

Let $w_1 = w_2 = w$.

Then from (17) we obtain

$$w^2 - 4w^2 = -r^4, \quad 2w^3 = \frac{r_1^8}{2h^2}. \tag{19}$$

From (19) follows: $h^* = \frac{\sqrt[4]{27}}{2\sqrt{2}} r_1$.

As $h^* > h$ from (16), then in this case two positive roots (15) will be multiple. Thus, when fulfilling condition (16), Eq. (15) will have two positive roots. And this means that Eq. (14) will have four material roots. Then Eq. (13) will have five real roots. This means that in this case in the wave picture there will be observed two loops on the travel-time curve and if $h = h^*$, the two loops will be coupled into one. This is supported by numerical calculations. Note that from general observations it follows that there can be an arbitrary number of loops on the travel-time curve for an arbitrary curvilinear boundary.

Let us analyze expression (7). In so doing let us consider a case of homogeneous waves. In this case each exponent in (7) will look like:

$$\mathrm{e}^{-\sqrt{\frac{\omega^2}{c_1^2} - k^2} d_2} \, \mathrm{e}^{\mathrm{i}\sqrt{\frac{\omega^2}{c_1^2} - k^2} d_1}. \tag{20}$$

In (20), there is an exponentially damping factor, describing a diffraction wave. Thus, the condition of limitedness of solution on infinity (the Sommerfeld condition) is now automatically fulfilled. The availability in the solution of a diffraction component results in the choice of a required solution. The number of real roots of the corresponding equation determines the wave picture and the number of loops on the travel-time curve. A complex distance describes the occurring diffraction effects. Finally, using the inverse Fourier transform we obtain the solution to (4) in the physical domain. Note that without difficulty this method is transferred to the initial statement (1)–(3) and other geophysical media. Absorption, for example, is taken into account according to [3,4]. In addition, it is possible to consider various distributed and concentrated sources such as the seismic momentum tensor [6].

4. Numerical calculations

Based on the proposed algorithm a program of calculation of complete wave fields for media with curvilinear boundaries has been developed. Such problems are considered in many publications. The

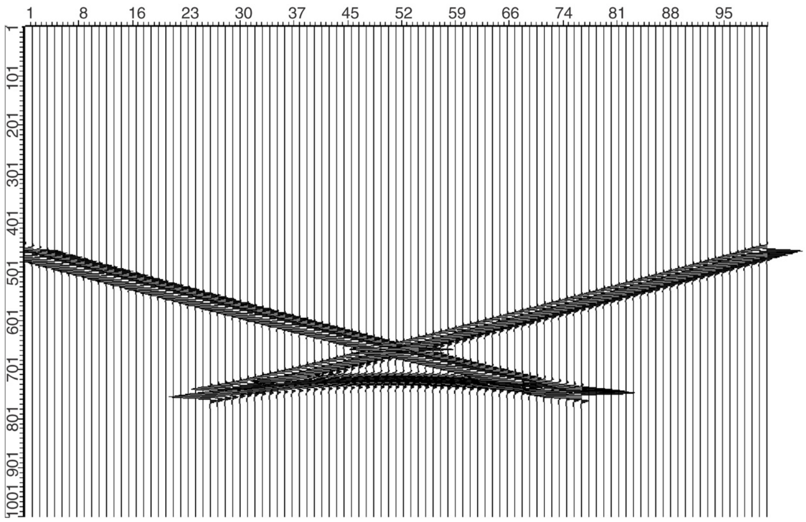

Fig. 2. Wave field for real thickness of a layer.

most widespread are asymptotic methods, including the ray method [7]. The research into extending the scope of application of this algorithm for domains with irregular fields is carried out [8]. Let us mention, in this connection, the Rayleigh method [9]. The latter has also a limited scope of application. The proposed method has no such limitations. It allows the calculation of wave fields for arbitrary curvilinear boundaries. Parabolic boundary (8) is selected as an illustration.

In Fig. 1, we can clearly distinguish a loop on the travel-time curve, on whose edges diffraction waves arise. These waves have rather a high, exponentially damping amplitude.

For comparison, Fig. 2 shows a similar picture, but with a real distance. It is seen that the diffraction waves have disappeared. The given calculation illustrates the wave picture in a “purely” ray case and coincides with the ray calculations [10]. The numerical simulation has shown that in the case of the boundary (12) the wave picture becomes more complicated. In this case there arise already two loops. The results of numerical simulation show that the wave picture for curvilinear boundaries becomes more complicated, especially, in the three-dimensional case. In addition to the diffraction effects, there can arise an arbitrary number of loops on the travel-time curve. In this connection, the correct approximation of a real boundary is of special importance.

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